数理解析研究所講究録 1414 巻 2005 年 8-14

On a subclass of n-starlike functions

Mugur Acu[1], Shigeyoshi Owa[2]

ABSTRACT. In 1999, S. Kanas and F. Ronning introduced the classes of functions starlike and convex, which are normalized whit $f(w) = f'(w)-1 = 0$ and w is a fixed point in U. In [1] the authors introduced the classes of functions close to convex and α-convex, which are normalized in the same way. All this definitions are somewhat similar to the ones for the uniformly type functions and is easy to see that for $w = 0$ are obtained the well-known classes of starlike, convex, close to convex and α-convex functions. In this paper we continue the investigation of the univalent functions normalized with $f(w) = f'(w) - 1 = 0$, where w is a fixed point in U.

2000 Mathematics Subject Classification: 30C45

Key words and phrases: Starlike functions, Briot-Bouquet differential subordination

1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$ and $S = \{f \in A : f \text{ is univalent in } U\}$.

We recall here the definitions of the well - known classes of starlike and convex functions:

$$S^* = \left\{ f \in A : Re \frac{zf'(z)}{f(z)} > 0 \, , \quad z \in U \right\},$$

$$S^c = \left\{ f \in A : Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0 \, , \quad z \in U \right\}$$

Let w be a fixed point in U and $A(w) = \{f \in \mathcal{H}(U) : f(w) = f'(w) - 1 = 0\}$.

In [3] S. Kanas and F. Ronning introduced the following classes:

$$S(w) = \{f \in A(w) : f \text{ is univalent in } U\}$$

$$ST(w) = S^*(w) = \left\{ f \in S(w) : Re \frac{(z-w)f'(z)}{f(z)} > 0 \, , \quad z \in U \right\}$$

$$CV(w) = S^c(w) = \left\{ f \in S(w) : 1 + Re \frac{(z-w)f''(z)}{f'(z)} > 0 \, , \quad z \in U \right\}.$$

It is obvious that exists a natural "Alexander relation" between the classes $S^*(w)$ and $S^c(w)$:

$$g \in S^c(w) \text{ if and only if } f(z) = (z-w)g'(z) \in S^*(w).$$

Let denote with $\mathcal{P}(w)$ the class of all functions $p(z) = 1 + \sum_{n=1}^{\infty} B_n \cdot (z-w)^n$ that are regular in U and satisfy $p(w) = 1$ and $Re\, p(z) > 0$ for $z \in U$.

2 Preliminary results

If is easy to see that a function $f_{(z)} \in A(w)$ have the series expansions:

$$f(z) = (z-w) + a_2(z-w)^2 + \dots$$

In [8] J. K. Wald gives the sharp bounds for the coefficients B_n of the function $p \in \mathcal{P}(w)$:

Teorema 2.1 *If $p(z) \in \mathcal{P}(w)$, $p(z) = 1 + \sum_{n=1}^{\infty} B_n \cdot (z-w)^n$, then*

(1) $$|B_n| \le \frac{2}{(1+d)(1-d)^n}\ , \quad \textit{where } d = |w| \textit{ and } n \ge 1.$$

Using the above result, S. Kanas and F. Ronning obtain in [3]:

Teorema 2.2 *Let $f \in S^*(w)$ and $f(z) = (z-w) + b_2(z-w)^2 + \dots$ Then*

(2) $$|b_2| \le \frac{2}{1-d^2}\ , \quad |b_3| \le \frac{3+d}{(1-d^2)^2}\ ,$$
$$|b_4| \le \frac{2}{3} \cdot \frac{(2+d)(3+d)}{(1-d^2)^3}\ , \quad |b_5| \le \frac{1}{6} \cdot \frac{(2+d)(3+d)(3d+5)}{(1-d^2)^4}$$

where $d = |w|$.

Remark 2.1 *It is clear that the above theorem also provides bounds for the coefficients of functions in $S^c(w)$, due to the relation between $S^c(w)$ and $S^*(w)$.*

In [1] are also defined the following sets:

$$D(w) = \left\{ z \in U :\ Re\left[\frac{w}{z}\right] < 1 \, and \, Re\left[\frac{z(1+z)}{(z-w)(1-z)}\right] > 0 \right\} for\, w \neq 0 \, and \, D(0) = U\,;$$

$$s(w) = \{f : D(w) \to \mathbb{C}\} \cap S(w)\,;\ s^*(w) = S^*(w) \cap s(w)$$

where w is a fixed point in U.

The authors consider the integral operator $L_a : A(w) \to A(w)$ defined by

$$f(z) = L_a F(z) = \frac{1+a}{(z-w)^a} \cdot \int_w^z F(t) \cdot (t-w)^{a-1} dt \,, \quad a \in \mathbb{R} \,, \quad a \geq 0. \tag{3}$$

The next theorem is results of the so called "admissible functions method" introduced by P. T. Mocanu and S. S. Miller (see [3], [4], [5]).

Teorema 2.3 *Let h convex in U and $Re[\beta h(z) + \gamma] > 0$, $z \in U$. If $p \in \mathcal{H}(U)$ with $p(0) = h(0)$ and p satisfied the Briot - Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \,, \quad \text{then } p(z) \prec h(z).$$

3 Main results

Definition 3.1 *Let w be a fixed point in U, $n \in \mathbb{N}$. We denote by D_w^n the differential operator:*

$$D_w^n : A(w) \to A(w) \text{ with :}$$
$$D_w^0 f(z) = f(z)$$
$$D_w^1 f(z) = D_w f(z) = (z-w) \cdot f'(z)$$
$$D_w^n f(z) = D_w \left(D_w^{n-1} f(z) \right) .$$

Remark 3.1 *For $f \in A(w)$, $f(w) = (z-w) + \sum_{j=2}^{\infty} a_j (z-w)^j$, we have*

$$D_w^n f(z) = (z-w) + \sum_{j=2}^{\infty} j^n \cdot a_j \cdot (z-w)^j .$$

It easy to see that if we take $w = 0$ we obtain the Sălăgean differential operator (see [7]).

Definition 3.2 *Let w be a fixed point in U, $n \in \mathbb{N}$ and $f \in S(w)$. We say that f is a n-w-starlike functions if*

$$Re \frac{D_w^{n+1} f(z)}{D_w^n f(z)} > 0 \,, \; z \in U .$$

We denote the class of all this functions by $S_n^*(w)$.

Remark 3.2 *1.* $S_0^*(w) = S^*(w)$ *and* $S_0^*(0) = S_n^*$*, where* S_n^* *is the class of n-starlike functions introduced by Sălăgean in [7].*
2. If $f(z) \in S_n^*(w)$ *and we denote* $D_w^n f(z) = g(z)$*, we obtain* $g(z) \in S^*(w)$.
3. Using the class $s(w)$*, we obtain* $s_n^*(w) = S_n^*(w) \cap s(w)$.

Teorema 3.1 *Let* w *be a fixed point in* U *and* $n \in \mathbb{N}$*. If* $f(z) \in s_{n+1}^*(w)$ *then* $f(z) \in s_n^*(w)$*. This means*

$$s_{n+1}^*(w) \subset s_n^*(w).$$

Proof. From $f(z) \in s_{n+1}^*(w)$ we have $Re \dfrac{D_w^{n+2} f(z)}{D_w^{n+1} f(z)} > 0\,,\ z \in U$.
We denote $p(z) = \dfrac{D_w^{n+1} f(z)}{D_w^n f(z)}$, where $p(0) = 1$ and $p(z) \in \mathcal{H}(U)$.
We obtain:

$$\frac{D_w^{n+2} f(z)}{D_w^{n+1} f(z)} = \frac{D_w\left(D_w^{n+1} f(z)\right)}{D_w\left(D_w^n f(z)\right)} =$$

$$= \frac{(z-w)\left(D_w^{n+1} f(z)\right)'}{(z=w)\left(D_w^n f(z)\right)'} = \frac{\left(D_w^{n+1} f(z)\right)'}{\left(D_w^n f(z)\right)'}$$

$$p'(z) = \frac{\left(D_w^{n+1} f(z)\right)' \cdot \left(D_w^n f(z)\right) - \left(D_w^{n+1} f(z)\right) \cdot \left(D_w^n f(z)\right)'}{\left(D_w^n f(z)\right)^2} =$$

$$= \frac{\left(D_w^{n+1} f(z)\right)'}{\left(D_w^n f(z)\right)'} \cdot \frac{\left(D_w^n f(z)\right)'}{D_w^n f(z)} - p(z) \cdot \frac{\left(D_w^n f(z)\right)'}{D_w^{n+} f(z)}.$$

Thus we have:

$$(z-w) \cdot p'(z) = \frac{\left(D_w^{n+1} f(z)\right)'}{\left(D_w^n f(z)\right)'} \cdot \frac{(z-w) \cdot \left(D_w^n f(z)\right)'}{D_w^n f(z)} - p(z) \cdot \frac{(z-w) \cdot \left(D_w^n f(z)\right)'}{D_w^n f(z)} =$$

$$(z-w) \cdot p'(z) = \frac{\left(D_w^{n+1} f(z)\right)'}{\left(D_w^n f(z)\right)'} \cdot p(z) - [p(z)]^2$$

and

$$\frac{\left(D_w^{n+1} f(z)\right)'}{\left(D_w^n f(z)\right)'} = p(z) + \frac{1}{p(z)} \cdot (z-w) \cdot p'(z).$$

From $Re \dfrac{D_w^{n+2} f(z)}{D_w^{n+1} f(z)} > 0$ we obtain $p(z) + \dfrac{1}{p(z)} \cdot (z-w) \cdot p'(z) \prec \dfrac{1+z}{1-z}$
or

$$p(z) + \frac{zp'(z)}{\dfrac{1}{1-\dfrac{w}{z}} \cdot p(z)} \prec \frac{1+z}{1-z} \equiv h(z)\,,\ with\ h(0) = 1\,.$$

From hypothesis we have $Re \left[\dfrac{1}{1-\dfrac{w}{z}} \cdot h(z)\right] > 0$, and thus from Theorem 2.3 we obtain $p(z) \prec h(z)$ or $Re\, p(z) > 0$. This means $f \in s_n^*(w)$.

Remark 3.3 *From Theorem 3.1 we obtain* $s_n^*(w) \subseteq s_0^*(w) \subseteq S^*(w)\,,\ n \in \mathbb{N}$.

Teorema 3.2 *If* $F(z) \in s_n^*(w)$ *then* $f(z) = L_a F(z) \in S_n^*(w)$, *where* L_a *is the integral operator defined by (3).*

Proof. From (3) we obtain

$$(1+a)\cdot F(z) = a\cdot f(z) + (z-w)\cdot f'(z)\,.$$

By means of the application of the operator D_w^{n+1} we obtain

$$(1+a)\cdot D_w^{n+1}F(z) = a\cdot D_w^{n+1}f(z) + D_w^{n+1}\left[(z-w)\cdot f'(z)\right]$$

or

$$(1+a)\cdot D_w^{n+1}F(z) = a\cdot D_w^{n+1}f(z) + D_w^{n+2}f(z)\,.$$

Similarly, by means of the application of the operator D_w^n we obtain

$$(1+a)\cdot D_w^{n}F(z) = a\cdot D_w^{n}f(z) + D_w^{n+1}f(z)\,.$$

Thus

$$\frac{D_w^{n+1}F(z)}{D_w^nF(z)} = \frac{\dfrac{D_w^{n+2}f(z)}{D_w^{n+1}f(z)}\cdot\dfrac{D_w^{n+1}f(z)}{D_w^nf(z)} + a\cdot\dfrac{D_w^{n+1}f(z)}{D_w^nf(z)}}{\dfrac{D_w^{n+1}f(z)}{D_w^nf(z)} + a}\,.$$

Using the notation $\dfrac{D_w^{n+1}f(z)}{D_w^nf(z)} = p(z)$, with $p(0)=1$, we have

$$\frac{(z-w)\cdot p'(z)}{p(z)} = \frac{D_w^{n+2}f(z)}{D_w^{n+1}f(z)} - p(z)$$

or

$$\frac{D_w^{n+2}f(z)}{D_w^{n+1}f(z)} = p(z) + \frac{(z-w)\cdot p'(z)}{p(z)}\,.$$

Thus

$$\frac{D_w^{n+1}F(z)}{D_w^nF(z)} = \frac{p(z)\left[p(z) + \dfrac{(z-w)p'(z)}{p(z)} + a\right]}{p(z)+a} =$$

$$= p(z) + \frac{zp'(z)}{\dfrac{1}{1-\dfrac{w}{z}}p(z) + \dfrac{a}{1-\dfrac{w}{z}}}\,.$$

From $F(z) \in s_n^*(w)$ we obtain $\dfrac{D_w^{n+1}F(z)}{D_w^nF(z)} \prec \dfrac{1+z}{1-z} \equiv h(z)$ or

$$p(z) + \frac{zp'(z)}{\dfrac{1}{1-\dfrac{w}{z}}p(z) + \dfrac{a}{1-\dfrac{w}{z}}} \prec h(z)\,.$$

From hypothesis we have $Re\left[\dfrac{1}{1-\dfrac{w}{z}} \cdot h(z) + \dfrac{a}{1-\dfrac{w}{z}}\right] > 0$ and from Theorem 2.3 we obtain $p(z) \prec h(z)$ or $Re\left\{\dfrac{D_w^{n+1}f(z)}{D_w^nf(z)}\right\} > 0\,,\ z \in U$. This means $f(z) = L_aF(z) \in S_n^*(w)$.

Remark 3.4 *If we consider $w = 0$ in Theorem 3.2 we obtain that the integral operator defined by (3) preserve the class of n-starlike functions, and if we consider $w = 0$ and $n = 0$ in the above Theorem we obtain that the integral operator defined by (3) preserve the well-known class of starlike functions.*

Teorema 3.3 *Let w be a fixed point in U and $f \in S_n^*(w)$ with $f(z) = (z-w) + \sum_{j=2}^{\infty} a_j \cdot (z-w)^j$. Then we have:*

$$|a_2| \le \frac{1}{2^{n-1} \cdot (1-d^2)}\,;$$

$$|a_3| \le \frac{3+d}{3^n \cdot (1-d^2)^2}\,;$$

$$|a_4| \le \frac{(2+d)(3+d)}{2^{2n-1} \cdot 3 \cdot (1-d^2)^3}\,;$$

$$|a_5| \le \frac{(2+d)(3+d)(3d+5)}{5^n \cdot 6 \cdot (1-d^2)^4}\,,$$

where $d = |w|$.

Proof. From Remark 3.2 for $f \in S_n^*(w)$ we obtain

$$D_w^nf(z) = g(z) \in S^*(w)\,. \tag{4}$$

If we consider $g(z) = (z-w) + \sum_{j=2}^{\infty} b_j \cdot (z-w)^j$, using Remark 3.1, from (4) we obtain $j^n \cdot a_j = b_j\,,\ j = 2, 3, \ldots$.

Thus we have $a_j = \dfrac{1}{j^n} \cdot b_j\,,\ j = 2, 3, \ldots$, and from the estimates (2) we get the result.

References

[1] M. Acu and S. Owa, *On some subclasses of univalent functions*, (to appear).

[2] A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math., LVIII (1991), 86 - 92.

[3] S. Kanas and F. Ronning, *Uniformly starlike and convex functions and other related classes of univelent functions*, Annales Univ. Mariae Curie - Sklodowska, Vol. L III, 10 (1999), 95 - 105.

[4] S. S. Miller and P. T. Mocanu, *Differential subordonations and univalent functions*, Mich. Math. 28 (1981), 157 - 171.

[5] S. S. Miller and P. T. Mocanu, *Univalent solution of Briot-Bouquet differential equations*, J. Differential Equations 56 (1985), 297 - 308.

[6] S. S. Miller and P. T. Mocanu, *On some classes of first-order differential subordinations*, Mich. Math. 32(1985), 185 - 195.

[7] Gr. Sălăgean, *Subclasses of univalent functions*, Complex Analysis. Fifth Roumanian-Finnish Seminar, Lectures Notes in Mathematics, 1013, Springer-Verlag, 1983, 362-372.

[8] J. K. Wald, *On starlike functions*, Ph. D. thesis, University of Delaware, Newark, Delaware (1978).

[1] University "Lucian Blaga" of Sibiu
Department of Mathematics
Str. Dr. I. Rațiu, No. 5-7
550012 - Sibiu, Romania
[2] Department of Mathematics
School of Science and Engineering
Kinki University
Higashi-Osaka, Osaka 577-8502, Japan